POWER SERIES REPRESENTATION OF PARTIAL DERIVATIVES REQUIRED IN ORBIT DETERMINATION

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1.0 INTRODUCTION

One of the prime difficulties of satellite data reduction lies in the computer time required to obtain the partial derivatives for estimating orbital parameters. If the problem is formulated in classical rectangular coordinates x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$, one is obliged to generate the 36 partial derivatives

$$
\begin{array}{cccccc}
\dfrac{\partial x}{\partial x_o} & \dfrac{\partial x}{\partial y_o} & \dfrac{\partial x}{\partial z_o} & \dfrac{\partial x}{\partial \dot{x}_o} & \dfrac{\partial x}{\partial \dot{y}_o} & \dfrac{\partial x}{\partial \dot{z}_o} \\
\dfrac{\partial y}{\partial x_o} & - & - & - & - & - \\
\vdots & & & & & \\
\dfrac{\partial \dot{z}}{\partial x_o} & - & - & - & - & \dfrac{\partial \dot{z}}{\partial \dot{z}_o}
\end{array}
$$

For perturbed artificial satellite motion, these partials cannot be obtained in closed analytic form. If they could be written in closed form, then one could write

$$x = f(x_o, y_o, z_o, \dot{x}_o, \dot{y}_o, \dot{z}_o, t - t_o)$$

(or a slight modification thereof if the chain rule is to be invoked) which would imply the artificial satellite problem was solved in closed analytic form. This appears to be impossible.

We are therefore restricted to computing the partials by numerical means. One approach is to hold five of the six parameters constant, vary the sixth, and numerically integrate to obtain the various rates of change. This method is very time consuming, but in some cases is the only possible way to obtain the desired result.

In the following sections, we describe a numerical integration method which gives the required partial derivatives as a by-product of the integration scheme.

2.0 THE POWER SERIES METHOD

One of the simplest means of numerical integration is doubtlessly that of Runge and Kutta. The procedure is adequate in most cases, but exhibits certain limitations on accuracy. For example, a 4th order Runge-Kutta scheme has an inherent error of the order of the 5th power of the step size. One may, of course, reduce the step size to increase accuracy, but this eventually leads to a trade-off with round-off error. To increase the accuracy beyond the minimum error obtained by determining an optimum trade-off between truncation and round-off, one must go to a whole new set of integration equations; for example, a 5th order Runge-Kutta scheme.

In this section, we will describe a numerical integration procedure in which only one input parameter must be changed in order to obtain a higher degree of accuracy.

2.1 General Concepts

Consider, for example, the differential equation

$$\dot{x} = f(x) \tag{1}$$

The method proposed here is called the "Power Series Solution." The basic idea is to assume a solution for equation (1) of the form

$$x = x_o + x_1 \Delta t + x_2 \Delta t^2 + \ldots x_n \Delta t^n = \sum_{i=0}^{n} x_i \Delta t^i \tag{2}$$

and determine, via recurrent formulae the unknown coefficients x_1, x_2, ... x_n. Equation (1) will yield one constant of integration, which we take as the initial condition, x_o.

Having all the coefficients of equation (2) we have then obtained a relation which represents the value of x within some interval of time, Δt.

We note that x_o is the position of the system when $\Delta t = 0$, and, by taking the time derivative of (2), that x_1 is the velocity at time $\Delta t = 0$.

The overall objective is then, having assumed some initial condition, say x_o, to obtain a recurrent relationship of the form

$$x_{i+1} = g(x_i, x_{i-1}, \dots x_o, i,) \qquad i \geqslant 0$$

so that we may compute, to any practical degree of accuracy, the value of x within some interval Δt. After choosing this particular Δt, a new value of x may be obtained by performing the Σ of equation (2) and taking this to be a new x_o. We then continue this process step by step until some value t_{FINAL} is reached.

The advantage of this process over that of the Runge-Kutta method is that if the assumed series converges, a higher order of accuracy may be obtained by simply computing more coefficients via the recurrence formula. There is no need to change the basic form of the integration scheme.

2.2 Example

Consider the equation

$$\dot{x} = x + 5 \tag{3}$$

We assume

$$x = \sum_{i=0}^{n} x_i \, \Delta t^i \tag{4}$$

so that

$$\dot{x} = \sum_{i=0}^{n-1} (i+1) \, x_{i+1} \, \Delta t^i$$

Substituting these last two equations into (3) yields,

$$\sum_{i=0}^{n-1} (i+1) \, x_{i+1} \, \Delta t^i = \left(\sum_{i=0}^{n} x_i \, \Delta t^i \right) + 5$$

or

$$\sum_{i=0}^{n-1} (i+1) \, x_{i+1} \, \Delta t^i = \left(\sum_{i=0}^{n-1} x_i \, \Delta t^i \right) + x_n \, \Delta t^n + 5$$

so that if we equate equal powers of Δt^i ($i = 0, 1, 2, \ldots n - 1$) one obtains

$$x_1 = x_o + 5 \qquad i = 0 \tag{5}$$

$$x_{i+1} = \frac{x_i}{i + 1} \qquad 1 \leq i < n \tag{6}$$

Once a value of x_o has been selected, the value of x_1 may be computed from (5) and all succeeding coefficients (x_2, x_3, ... x_{n-1}) may be computed from (6).

2.3 Convergence

We now address ourselves to the question of the convergence of the series solution (4).

If we let

$$S_n = \sum_{i=0}^{n} x_i \Delta t^i$$

we recall that if

$$S = \operatorname*{Limit}_{n \to \infty} S_n$$

exists, the series is said to converge to the value S.

The following theorems will be used to answer the question of convergence.

THEOREM I

The geometric series

$$\sum_{i=0}^{\infty} a \, \Delta^{i} \qquad (a = \text{const})$$

converges for $|\Delta| < 1$.

THEOREM II

If

$$\sum_{i=0}^{\infty} |x_i| \, |\Delta|^{i}$$

converges, then

$$\sum_{i=0}^{\infty} x_i \, \Delta^{i}$$

also converges.

In order to prove the convergence of the series (4), we introduce ε and $\bar{x}$ as arbitrary, positive, finite numbers. We wish to show that the inequality

$$|x_i| \leq \bar{x}\,\varepsilon^i \tag{7}$$

implies the validity of the inequality

$$|x_{i+1}| \leq \bar{x}\,\varepsilon^{i+1} \tag{8}$$

That is, we will show that all values of $|x_i|$ ($i = 1, 2, 3, \ldots$) are bounded provided x_o is finite. Theorems I and II then will be invoked to prove the convergence of the series (4).

To start the proof of convergence, we write (6) as

$$|x_{i+1}| = \frac{|x_i|}{|i+1|} = \frac{|x_i|}{i+1}$$

Substituting equation (7) into the above results in

$$|x_{i+1}| \leq \frac{\bar{x}\varepsilon^i}{i+1} \tag{9}$$

Substitution of (8) for the left-hand side of (9) results in either

$$\overline{x}\varepsilon^{i+1} \leq \frac{\overline{x}\varepsilon^{i}}{i+1} \tag{10}$$

or

$$\overline{x}\varepsilon^{i+1} \leq \frac{\overline{x}\varepsilon^{i}}{i+1} \tag{11}$$

For the sake of clarity in that which follows, we rewrite (9), (10), and (11) as

$$|x_{i+1}| \leq A \tag{12}$$

$$\overline{x}\varepsilon^{i+1} \leq A \tag{13}$$

$$\overline{x}\varepsilon^{i+1} \geq A \tag{14}$$

Equations (12) and (13) yield no significant information concerning a relationship between $|x_{i+1}|$ and $\overline{x}\varepsilon^{i+1}$. However, (12) and (14) demand that

$$|x_{i+1}| \leq \overline{x}\varepsilon^{i+1}$$

so that a sufficient condition that (8) holds true is (14), and therefore,

$$\bar{x}\varepsilon^{i+1} \geq \frac{\bar{x}\varepsilon^{i}}{i+1}$$

which we write as

$$\bar{x}\varepsilon^{i} \left(\varepsilon - \frac{1}{i+1}\right) \geq 0$$

Since it was assumed that $\bar{x}$ and ε were positive and non-zero, we must have

$$\varepsilon \geq \frac{1}{i+1}$$

But, for any $i \geqslant 1$,

$$\frac{1}{i+1} \leq \frac{1}{2} \qquad (i \geq 1)$$

which puts a lower bound on ε, i.e., we shall take

$$\varepsilon \geq \frac{1}{2}$$

We have now proved that all x_i are bounded provided

$$\varepsilon \geq \frac{1}{2}$$

If we define x* as the infinite series of absolute values of the solution series (4), then

$$\begin{aligned} x^* &= |x_o| + |x_1|\,|\Delta t| + \ldots |x_n|\,|\Delta t^n| + \ldots \\ &= \sum_{i=0}^{\infty} |x_i|\,|\Delta t|^i \end{aligned} \tag{15}$$

Theorem II states, however, that if (15) can be shown to be convergent, then

$$x = \sum_{i=0}^{\infty} x_i \quad \Delta t^i$$

is also convergent. Thus, the burden of the proof lies in showing that (15) is convergent. To this end, we substitute (7) into (15) to obtain

$$x^* \leq \sum_{i=0}^{\infty} \quad \bar{x}\, \varepsilon^i\, |\Delta t|^i$$

which, by Theorem I is convergent in the disk

$$|\Delta t| < \frac{1}{\varepsilon}$$

We have now proved convergence for the series

$$x = \sum_{i=0}^{\infty} x_i \, \Delta t^i$$

provided

$$|\Delta t| < \frac{1}{\varepsilon}$$

and

$$\varepsilon \geq \frac{1}{2}$$

2.4 Practical Applications

The biggest asset of the power series solution lies in the practical application of the method. That is, one need not waste time on useless accuracy. For example, assuming the series converges and the input data is accurate to some β, we now choose some value of "n" in equation (6) and compute x_1, x_2, ... x_{n-1}.

Experience has shown that $\sum_{j=i+1}^{\infty} x_j \Delta t^j$ is practically always $\leq x_i \Delta t^i$, and therefore, the stepsize may be computed from,

$$|x_{n-1}| \, \Delta t^{n-1} = \beta$$

or

$$\Delta t = \left[\frac{\beta}{|x_{n-1}|} \right]^{\frac{1}{n-1}} \tag{16}$$

Computing the integration step in this manner guarantees a certain degree of accuracy (assuming round-off doesn't become uncontrolled due to very long integration times), and at the same time minimizes the computer time for some prescribed number of coefficients. Of course, a different value of "n" may also decrease the time required.

The process of interpolation also has advantages over many of the other methods. In the power series approach, the interpolating polynomial is already available ($x = \sum_{i=0}^{\infty} x_i \, \Delta t^i$) and one need only to substitute some value of Δt to obtain x at any time, say Δt^*, where $-\Delta t < \Delta t^* < \Delta t$. The results will be at least as accurate as the case where the step size is Δt.

Inverse interpolation is also obtained by simply inverting the polynomial to yield a value of Δt^* for some given x. Since the polynomial can be inverted by iterative methods, the results can be at least as accurate as the integration itself.

3.0 EVALUATION OF PARTIAL DERIVATIVES

We now address ourselves to the problem of obtaining expressions for the partial derivatives given in section 1.0.

Many programs simply choose small increments in the orbital parameters (Δx, Δy, Δz, $\Delta\dot{x}$, $\Delta\dot{y}$, $\Delta\dot{z}$), add these increments to the original parameters (one at a time) and numerically integrate to obtain, for example

$$\frac{\partial x_t}{\partial x_o} = \frac{x_t(x_o+\Delta x, y_o, z_o, \dot{x}_o, \dot{y}_o, \dot{z}_o) - x_t(x_o, y_o, z_o, \dot{x}_o, \dot{y}_o, \dot{z}_o)}{\Delta x}$$

This procedure is very time consuming on a computer, and should be discarded if possible.

In the next section we describe a procedure for computing partial derivatives which is primarily a by-product of the integration scheme just discussed.

3.1 General Concepts

We start by considering the simple example of section 2.2. The assumed series solution was (equation 2)

$$x = x_o + x_1\,\Delta t + x_2\,\Delta t^2 + \ldots\ x_n\,\Delta t^n$$

Equations (5) and (6) show that we may write this last equation as

$$x = x_o + x_1(x_o)\Delta t + x_2(x_o)\Delta t^2 + \ldots x_n(x_o)\Delta t^n \tag{17}$$

so that

$$\frac{\partial x}{\partial x_o} = 1 + \frac{\partial x_1(x_0)}{\partial x_o}\Delta t + \frac{\partial x_2(x_0)}{\partial x_o}\Delta t^2 \text{ --------} \tag{18}$$

This last equation represents $\partial x/\partial x_o$ anywhere in the domain of convergence for Δt. All that is required is to compute the partials $\partial x_1/\partial x_o$, $\partial x_2/\partial x_o$, --- $\partial x_n/\partial x_o$, which are constants in the domain for Δt. From equations (5) and (6)

$$\frac{\partial x_1}{\partial x_o} = 1 \tag{19}$$

$$\frac{\partial x_{i+1}}{\partial x_o} = \frac{1}{i+1} \frac{\partial x_i}{\partial x_o} \tag{20}$$

The coefficients $\partial x_i/\partial x_o$ (i = 1, 2, --- n) may be computed by (19) and (20) and hence, we now have only to choose a Δt and perform the summation of equation (18) to obtain $\partial x/\partial x_o$ at any time in the domain of convergence for Δt. Notice that it is not necessary to reevaluate the partial $\partial x_i/\partial x_o$ (i = 1, 2, --- n) for every different time in the interval Δt.

The advantages of this method become apparent with a little examination of the physical interpretation of (18). For short arc orbit determination (15-20 minutes) it appears that if we assume a reasonable upper limit on the number of coefficients x_n (say $n < 30$), then only one evaluation of the various coefficient $\partial x_i/\partial x_o$ (i = 1, 30) is required. The partials at any time in the domain of convergence may then be computed by simply assigning the required value to Δt and summing as shown in (18). This appears to be many times faster than other techniques used to get the required partial derivatives.

3.2 Convergence

The proof of convergence for (18) is identical to that given for (2). Only the high-lights will be given here. Again, let ε and $\overline{x}$ be arbitrary, positive, finite numbers. We wish to show that the inequality

$$\left|\frac{\partial x_i}{\partial x_o}\right| \leq \overline{x}\varepsilon^i \tag{21}$$

implies the validity of the inequality

$$\left| \frac{\partial x_{i+1}}{\partial x_o} \right| \leq \bar{x}\varepsilon^{i+1} \tag{22}$$

for $i \geq 1$. From 20

$$\left| \frac{\partial x_{i+1}}{\partial x_o} \right| \leq \frac{1}{i+1} \quad \bar{x}\varepsilon^{i} \tag{23}$$

Substituting (22) into (23) gives the <u>sufficient</u> condition

$$\bar{x}\varepsilon^{i+1} \geq \frac{\bar{x}\varepsilon^{i}}{i+1} \tag{24}$$

From this point on the proof is identical to that given in section 2.3, and will not be duplicated.

4.0 APPLICATION TO THE TWO-BODY ARTIFICIAL SATELLITE PROBLEM

The following four sections are devoted to the analytical development of the recursive relations for the two-body problem. While such an application may appear to be a waste of time, this intermediate study will yield a feel of the method; its virtues, its pitfalls. It is also desirable to apply a new and untried method to a problem with a known closed form solution, since this aids in technical evaluation and accuracy checks.

The primary concern in these sections is to determine how fast the coefficients converge for the assumed series solution and the series for the partial derivatives.

4.1 The Integration Formulae

The equations of motion for the two body problem can be put in the form

$$\ddot{x} = \frac{-\mu x}{r^3} \qquad (25a)$$

$$\ddot{y} = \frac{-\mu y}{r^3} \qquad (25b)$$

$$\ddot{z} = \frac{-\mu z}{r^3} \qquad (25c)$$

where

$$r^2 = x^2 + y^2 + z^2$$

$$\mu = \text{constant} \tag{25d}$$

Define

$$A_1 = \frac{1}{r^3}$$

or

$$A_1 r^3 = 1$$

which yields

$$3A_1\dot{r} + r\dot{A}_1 = 0 \tag{26}$$

Equations (25) and (26) can now be written in the following form:

$$r\dot{r} = x\dot{x} + y\dot{y} + z\dot{z} \tag{27a}$$

$$r\dot{A}_1 = -3A_1\dot{r} \tag{27b}$$

$$\ddot{x} = -\mu A x \tag{27c}$$

$$\ddot{y} = -\mu A y \tag{27d}$$

$$\ddot{z} = -\mu A z \tag{27e}$$

Assume the solutions

$$x = \sum_{i=0}^{m} x_i \ \Delta t^i \tag{28a}$$

$$y = \sum_{i=0}^{m} y_i \ t^i \tag{28b}$$

$$z = \sum_{i=0}^{m} z_i \ \Delta t^i \tag{28c}$$

$$r = \sum_{i=0}^{m} r_i \ \Delta t^i \tag{28d}$$

$$A_1 = \sum_{i=0}^{m} a_{1,i} \ \Delta t^i \tag{28e}$$

From these we obtain, to the same powers of Δt,

$$\dot{x} = \sum_{i=0}^{m} (i+1)\, x_{i+1}\, \Delta t^{i} \tag{29a}$$

$$\dot{y} = \sum_{i=0}^{m} (i+1)\, y_{i+1}\, \Delta t^{i} \tag{29b}$$

$$\dot{z} = \sum_{i=0}^{m} (i+1)\, z_{i+1}\, \Delta t^{i} \tag{29c}$$

$$\dot{r} = \sum_{i=0}^{m} (i+1)\, r_{i+1}\, \Delta t^{i} \tag{29d}$$

$$\dot{A}_1 = \sum_{i=0}^{m} (i+1)\, a_{1,i+1}\, \Delta t^{i} \tag{29e}$$

and therefore

$$\ddot{x} = \sum_{i=0}^{m} (i+1)\,(i+2)\, x_{i+2}\, \Delta t^{i} \tag{30a}$$

$$\ddot{y} = \sum_{i=0}^{m} (i+1)\,(i+2)\, y_{i+2}\, \Delta t^{i} \tag{30b}$$

$$\ddot{z} = \sum_{i=0}^{m} (i+1)\,(i+2)\, z_{i+2}\, \Delta t^{i} \tag{30c}$$

We now wish to obtain recurrent formulae for the unknown coefficients x_i, y_i, z_i , r_i, and $a_{1,i}$. Note that the equations to be solved are second order, and hence six constants of integration are at our disposal. Choose these six constants to be x_0, y_0, z_0, x_1, y_1, and z_1 (which correspond to the position and velocity at time $\Delta t = 0$).

In order to obtain the recurrence formulae, we utilize the following general formulae:

$$\left(\sum_{i=0}^{m} p_i \;\Delta t^i\right)\left(\sum_{n=0}^{m} q_n \;\Delta t^n\right) = \sum_{n=0}^{m}\left(\sum_{i=0}^{n} p_i \; q_{n-i}\right)\Delta t^n + \sum_{\nu=1}^{m}\left(\sum_{i=0}^{m-\nu} p_{i+\nu} \; q_{m-i}\right) \Delta t^{m+\nu} \tag{31a}$$

$$\left(\sum_{i=0}^{m} (i+1)\; p_{i+1} \;\Delta t^i\right)\left(\sum_{n=0}^{m} q_n \;\Delta t^n\right) = \sum_{n=0}^{m}\left(\sum_{i=0}^{n} (i+1)\; p_{i+1}\; q_{n-i}\right) \Delta t^n + \sum_{\nu=1}^{m}\left(\sum_{i=0}^{m-\nu} (i+\nu+1)\; p_{i+\nu+1}\; q_{m-i}\right) \Delta t^{m+\nu} \tag{31b}$$

Consider equation (27a)

$$r\dot{r} = x\dot{x} + y\dot{y} + z\dot{z}$$

From (31b)

$$r\dot{r} = \left(\sum_{i=0}^{m} (i+1)\ r_{i+1}\ \Delta t^{i}\right)\left(\sum_{n=0}^{m} r_n\ \Delta t^{n}\right) =$$

$$\sum_{n=0}^{m}\left(\sum_{i=0}^{n} (i+1)\ r_{i+1}\ r_{n-i}\right)\Delta t^{n} + \Big[\ \Big]_1\ \Delta t^{m+\nu}$$

where $\Big[\ \Big]_1\ \Delta t^{m+\nu}$ represents some quantity times $\Delta t^{m+\nu}$.

Similar expressions exist for $x\dot{x}$, $y\dot{y}$, and $z\dot{z}$. Substitution of these results into equation (27a) gives

$$\left[\sum_{n=0}^{m}\left(\sum_{i=0}^{n} (i+1)\ r_{i+1}\ r_{n-i}\right)\Delta t^{n}\right] + \Big[\ \Big]_1\ \Delta t^{m+\nu} =$$

$$\left[\sum_{n=0}^{m}\left(\sum_{i=0}^{m} (i+1)\ x_{i+1}\ x_{n-i}\right)\Delta t^{n}\right] + \Big[\ \Big]_2\ \Delta t^{m+\nu}$$

$$+\left[\sum_{n=0}^{m}\left(\sum_{i=0}^{n} (i+1)\ y_{i+1}\ y_{n-i}\right)\Delta t^{n}\right] + \Big[\ \Big]_3\ \Delta t^{m+\nu}$$

$$+\left[\sum_{n=0}^{m}\left(\sum_{i=0}^{n} (i+1)\ z_{i+1}\ z_{n-i}\right)\Delta t^{n}\right] + \Big[\ \Big]_4\ \Delta t^{m+\nu}$$

which we write as

$$\sum_{n=0}^{m} \left(\sum_{i=0}^{n} (i+1)\, r_{i+1}\, r_{n-i} \right) \Delta t^{n} =$$

$$\left[\sum_{n=0}^{m} \left(\sum_{i=0}^{n} (i+1)\, (x_{i+1}\, x_{n-i} + y_{i+1}\, y_{n-i} \right.\right.$$

$$\left.\left. + \; z_{i+1}\, z_{n-i} \right) \Delta t^{n} \right] + \left[\; \right]_5 \Delta t^{m+\nu}$$

Since we have assumed only "m" terms in the series solution, all coefficients of $\Delta t^{m+\nu}$ can be assumed to be "zero." Equating then, coefficients of Δt^{n}, we obtain

$$\sum_{i=0}^{n} (i+1)\, r_{i+1}\, r_{n-i} =$$

$$\sum_{i=0}^{n} (i+1) \left[x_{i+1}\, x_{n-i} + y_{i+1}\, y_{n-i} + z_{i+1}\, z_{n-i} \right]$$

or

$$(n+1)\, r_0\, r_{n+1} + \sum_{i=0}^{n-1} (i+1)\, r_{i+1}\, r_{n-i}$$

$$= (n+1)\, (x_{n+1}\, x_0 + y_{n+1}\, y_0 + z_{n+1}\, z_0)$$

$$+ \sum_{i=0}^{n-1} (i+1)\, (x_{i+1}\, x_{n-i} + y_{i+1}\, y_{n-i} + z_{i+1}\, z_{n-i})$$

which we write as

$$(n+1)\, r_0\, r_{n+1} = (n+1)\, (x_{n+1}\, x_0 + y_{n+1}\, y_0 + z_{n+1}\, z_0)$$

$$+ \sum_{i=0}^{n-1} (i+1)\, (x_{i+1}\, x_{n-i} + y_{i+1}\, y_{n-i} \qquad (32a)$$

$$+ z_{i+1}\, z_{n-i} - r_{i+1}\, r_{n-i})$$

which is the desired recurrence formula for the r_i. In an identical manner, one obtains the following recurrence formulae for equations (27b), (27c), (27d) and (27e):

$$(n+1)\, r_0\, a_{1,n+1} = -3(n+1)\, r_{n+1}\, a_{1,0}$$

$$- \sum_{i=0}^{n-1} (i+1)\, (3r_{i+1}\, a_{1,n-i} + a_{1,i+1}\, r_{n-i}) \qquad (32b)$$

$$(n+1)\,(n+2)\, x_{n+2} = -\mu \sum_{i=0}^{n} a_{1,i}\, x_{n-i} \qquad (32c)$$

$$(n+1)\,(n+2)\, y_{n+2} = -\mu \sum_{i=0}^{n} a_{1,i}\, y_{n-i} \qquad (32d)$$

$$(n+1)\,(n+2)\, z_{n+2} = -\mu \sum_{i=0}^{n} a_{1,i}\, z_{n-i} \qquad (32e)$$

4.2 The Partial Derivative Formulae

The required partial derivatives were given in section 1.0, and the method of obtaining them follows the ideas laid down in section 3.1.

Recalling that x_0, y_0, and z_0 represent the position at time $\Delta t = 0$ and x_1, y_1, and z_1 the velocity at time $\Delta t = 0$, we write

$$\frac{\partial x_0}{\partial x_0} = \frac{\partial x_1}{\partial x_1} = \frac{\partial y_0}{\partial y_0} = \frac{\partial y_1}{\partial y_1} = \frac{\partial z_0}{\partial z_0} = \frac{\partial z_1}{\partial z_1} = 1$$

In the interest of compactness, we put this last result in the following form:

$$\frac{\partial x^*}{\partial x^*} = 1 \quad \Big\{ \; x^* = x_0,\, y_0,\, z_0,\, x_1,\, y_1,\, z_1 \tag{33a}$$

Also, since the position and velocity components at time $\Delta t = 0$ are independent of each other, we have

$$\frac{\partial x^*}{\partial x^{**}} = 0 \left\{ \begin{array}{l} x^* = x_0,\, y_0,\, z_0,\, x_1,\, y_1,\, z_1 \\ \\ x^{**} = x_0,\, y_0,\, z_0,\, x_1,\, y_1,\, z_1 \end{array} \right. \qquad x^* \neq x^{**} \tag{33b}$$

Let us now define coefficients of Δt^m to be of the m^{th} order. The partials of "zero" order in addition to those of equations (33) are

$$\frac{\partial r_0}{\partial x^*} \; ; \; \frac{\partial a_{1,0}}{\partial x^*} \quad \Big\{ \; x^* = x_0,\, y_0,\, z_0,\, x_1,\, y_1,\, z_1$$

The partials of r_0 can be obtained from

$$r_0^2 = x_0^2 + y_0^2 + z_0^2$$

which gives

$$\frac{\partial r_0}{\partial x^*} = \begin{cases} \dfrac{x^*}{r_0} & x^* = x_0,\ y_0,\ z_0 \\ 0 & x^* = x_1,\ y_1,\ z_1 \end{cases} \tag{34}$$

For $a_{1,0}$

$$a_{1,0} = \frac{1}{r_0^3}$$

which yields

$$\frac{\partial a_{1,0}}{\partial x^*} = \begin{cases} -\dfrac{3a_{1,0}}{r_0}\ \dfrac{\partial r_0}{\partial x^*} & x^* = x_0,\ y_0,\ z_0 \\ 0 & x^* = x_1,\ y_1,\ z_1 \end{cases} \tag{35}$$

This completes all zero ordered partials.

FIRST ORDER PARTIALS

Equations (33) give some of the first order partials. The remaining are

$$\frac{\partial r_1}{\partial x^*} \;;\quad \frac{\partial a_{1,1}}{\partial x^*} \qquad x^* = x_0,\ y_0,\ z_0,\ x_1,\ y_1,\ z_1$$

Setting n = 0 in (32a) gives

$$r_0 r_1 = x_1 x_0 + y_1 y_0 + z_1 z_0$$

or

$$\frac{\partial r_1}{\partial x^*} = \frac{1}{r_0}\left[x_1 \frac{\partial x_0}{\partial x^*} + x_0 \frac{\partial x_1}{\partial x^*} + y_1 \frac{\partial y_0}{\partial x^*}\right.$$

$$\left. y_0 \frac{\partial y_1}{\partial x^*} + z_1 \frac{\partial z_0}{\partial x^*} + z_0 \frac{\partial z_1}{\partial x^*} - r_1 \frac{\partial r_0}{\partial x^*}\right] \tag{36}$$

$$x^* = x_0,\ y_0,\ z_0,\ x_1,\ y_1,\ z_1$$

It is obvious that several of the terms of (36) are zero for a given x*. However, in the interest of uniformity and simplicity of programming, we leave this last equation in its present form so that one "DO" loop can be developed for all partials defined by (36). This trend will be carried on throughout the analysis.

Setting n = 0 in (32b) gives

$$r_0 a_{1,1} = -3 r_1 a_{1,0}$$

or

$$\frac{\partial a_{1,1}}{\partial x^*} = -\frac{1}{r_0}\left[3\left(r_1 \frac{\partial a_{1,0}}{\partial x^*} + a_{1,0}\frac{\partial r_1}{\partial x^*}\right) + a_{1,1}\frac{\partial r_0}{\partial x^*}\right] \tag{37}$$

Equations (36) and (37) define all remaining first order partials.

n^{th} ORDER PARTIALS

All higher order partials may be computed by taking the partials of the recursive equations (32a) - (32e). In all cases, x* is to be replaced with x_0, y_0, z_0, x_1, y_1, and z_1.

For (32a)

$$
\begin{aligned}
(n+1)\, r_0 \frac{\partial r_{n+1}}{\partial x^*} = (n+1) \Bigg[& x_{n+1} \frac{\partial x_0}{\partial x^*} + x_0 \frac{\partial x_{n+1}}{\partial x^*} \\
& + y_{n+1} \frac{\partial y_0}{\partial x^*} + y_0 \frac{\partial y_{n+1}}{\partial x^*} + z_{n+1} \frac{\partial z_0}{\partial x^*} \\
& + z_0 \frac{\partial z_{n+1}}{\partial x^*} - r_{n+1} \frac{\partial r_0}{\partial x^*} \Bigg] \\
& + \sum_{i=0}^{n-1} (i+1) \Bigg\{ x_{i+1} \frac{\partial x_{n-i}}{\partial x^*} + x_{n-i} \frac{\partial x_{i+1}}{\partial x^*} \\
& + y_{i+1} \frac{\partial y_{n-i}}{\partial x^*} + y_{n-i} \frac{\partial y_{i+1}}{\partial x^*} \\
& + z_{i+1} \frac{\partial z_{n-i}}{\partial x^*} + z_{n-i} \frac{\partial z_{i+1}}{\partial x^*} \\
& - r_{i+1} \frac{\partial r_{n-i}}{\partial x^*} - r_{n-i} \frac{\partial r_{i+1}}{\partial x^*} \Bigg\}
\end{aligned} \tag{38}
$$

For (32b)

$$(n+1) \quad r_0 \frac{\partial a_{1,n+1}}{\partial x^*} = -(n+1)\left[a_{1,n+1} \frac{\partial r_0}{\partial x^*}\right.$$

$$\left. + 3\left(r_{n+1} \frac{\partial a_{1,0}}{\partial x^*} + a_{1,0} \frac{\partial r_{n+1}}{\partial x^*}\right)\right]$$

$$- \sum_{i=0}^{n-1} (i+1) \left\{ 3 \left(r_{i+1} \frac{\partial a_{1,n-i}}{\partial x^*} \right)\right. \tag{39}$$

$$+ a_{1,n-i} \frac{\partial r_{i+1}}{\partial x^*}$$

$$\left. + a_{1,i+1} \frac{\partial r_{n-i}}{\partial x^*} + r_{n-i} \frac{\partial a_{1,i+1}}{\partial x^*} \right\}$$

For (32c)

$$\frac{\partial x_{n+2}}{\partial x^*} = \frac{-\mu}{(n+1)(n+2)} \sum_{i=0}^{n} \left(a_{1,i} \frac{\partial x_{n-i}}{\partial x^*} \right.$$

$$\left. + x_{n-i} \frac{\partial a_{1,i}}{\partial x^*} \right) \tag{40}$$

For (32d)

$$\frac{\partial y_{n+2}}{\partial x^*} = \frac{-\mu}{(n+1)(n+2)} \sum_{i=0}^{n} \left(a_{1,i} \frac{\partial y_{n-i}}{\partial x^*} + y_{n-i} \frac{\partial a_{1,i}}{\partial x^*} \right) \tag{41}$$

For (32e)

$$\frac{\partial z_{n+2}}{\partial x^*} = \frac{-\mu}{(n+1)(n+2)} \sum_{i=0}^{n} \left(a_{1,i} \frac{\partial z_{n-i}}{\partial x^*} + z_{n-i} \frac{\partial a_{1,i}}{\partial x^*} \right) \tag{42}$$

4.3 Convergence

In this section we wish to prove convergence for the assumed series provided x_0, y_0, z_0, x_1, y_1, z_1 are bounded and $r_0 \neq 0$. We start with the series for the integration scheme.

Define $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{r}$, $\bar{a}$, and ε as arbitrary positive finite numbers. Furthermore, define for all $n \geq 2$

$$k_n = \frac{1}{n(n+1)} \tag{43}$$

To prove convergence, we want to show that for any $n \geq 2$, that the inequalities

$$|x_n| \leq \bar{x}\, k_n\, \varepsilon^n \tag{44a}$$

$$|y_n| \leq \bar{y}\, k_n\, \varepsilon^n \tag{44b}$$

$$|z_n| \leq \bar{z}\, k_n\, \varepsilon^n \tag{44c}$$

$$|r_n| \leq \bar{r}\, k_n\, \varepsilon^n \tag{44d}$$

$$|a_{1,n}| \leq \bar{a}_1\, k_n\, \varepsilon^n \tag{44e}$$

imply the validity of

$$| x_{n+1} | \leq \bar{x} \, k_{n+1} \, \varepsilon^{n+1} \tag{45a}$$

$$| y_{n+1} | \leq \bar{y} \, k_{n+1} \, \varepsilon^{n+1} \tag{45b}$$

$$| z_{n+1} | \leq \bar{z} \, k_{n+1} \, \varepsilon^{n+1} \tag{45c}$$

$$| r_{n+1} | \leq \bar{r} \, k_{n+1} \, \varepsilon^{n+1} \tag{45d}$$

$$|a_{1,n+1}| \leq \bar{a}_1 \, k_{n+1} \, \varepsilon^{n+1} \tag{45e}$$

That is, we wish to show that all x_n, y_n, z_n, r_n, and a_n are bounded provided the initial conditions are finite and $r_0 \neq 0$. The theorems of section 2.3 may then be invoked to complete the proof of convergence.

We start with equation (32a), which we write as

$$(n+1) \quad r_0 \quad |r_{n+1}| \leq (n+1) \Big\{ |x_{n+1}| \quad |x_0| + |y_{n+1}| \quad |y_0| + |z_{n+1}| \quad |z_0| \Big\}$$

(cont. next page)

$$+ \sum_{i=0}^{n-1} (i+1) \left\{ |x_{i+1}| \; |x_{n-i}| + |y_{i+1}| \; |y_{n-i}| \right.$$

$$\left. + |z_{i+1}| \; |z_{n-i}| + |r_{i+1}| \; |r_{n-i}| \right\}$$

Substituting (44) into this last result gives

$$(n+1) \; r_0 \; |r_{n+1}| \leq (n+1) \; k_{n+1} \; \varepsilon^{n+1} \left\{ \bar{x} \; |x_0| \right.$$

$$\left. + \bar{y} \; |y_0| + \bar{z} \; |z_0| \right\}$$

$$+ (\bar{x}^2 + \bar{y}^2 + \bar{z}^2 + \bar{r}^2) \; \varepsilon^{n+1} \sum_{i=0}^{n-1} (i+1) \; k_{n-i} \; k_{i+1}$$

which we write in abbreviated form as

$$(n+1) \; r_0 \; |r_{n+1}| \leq P \tag{46a}$$

Substituting (45d) into the last equation gives either

$$(n+1) \quad r_0 \quad \bar{r} \quad k_{n+1} \quad \varepsilon^{n+1} \quad < \quad P \tag{46b}$$

or

$$(n+1) \quad r_0 \quad \bar{r} \quad k_{n+1} \quad \varepsilon^{n+1} \quad \geq \quad P \tag{46c}$$

Equations (46a) and (46b) give no information concerning a relationship between r_{n+1}, k_{n+1}, and ε^{n+1}. But (46a) and (46c) demand that

$$|r_{n+1}| \quad \leq \quad \bar{r} \quad k_{n+1} \quad \varepsilon^{n+1}$$

and therefore a <u>sufficient</u> condition that (45d) hold true is (46c). Substituting (43) into (46c) gives

$$r_0 \quad \bar{r} \quad \geq \quad \bar{x} \quad |x_0| \quad + \quad \bar{y} \quad |y_0| \quad + \quad \bar{z} \quad |z_0| \\ + \quad \left[\bar{x}^2 + \bar{y}^2 + \bar{z}^2 + \bar{r}^2 \right] \quad (n+2) \quad \sum_{i=0}^{n-1} (i+1) \; k_{i+1} \; k_{n-i} \tag{47}$$

We now want to obtain some kind of limit on

$$(n+2) \sum_{i=0}^{n-1} (i+1) \; k_{i+1} \; k_{n-i}$$

Writing

$$(n+2) = (i+2) \; (n-i+1) - (i+1) \; (n-i)$$

then

$$(n+2) \; k_{i+1} \; k_{n-i} = \frac{1}{n+1} \left(\frac{1}{i+1} + \frac{1}{n-i} \right)$$

$$- \frac{1}{n+3} \left(\frac{1}{i+2} + \frac{1}{n-i+1} \right)$$

or

$$(n+2) \sum_{i=0}^{n-1} (i+1) \; k_{i+1} \; k_{n-i}$$

$$= \sum_{i=0}^{n-1} \left[\frac{i+1}{n+1} \left(\frac{1}{i+1} + \frac{1}{n-i} \right) - \frac{i+1}{n+3} \left(\frac{1}{i+2} + \frac{1}{n-i+1} \right) \right]$$

$$= \left(\sum_{i=0}^{n-1} \frac{1}{n-i} \right) - \sum_{i=0}^{n-1} \frac{i+1}{(i+2)(n-i+1)}$$

Using partial fractions for the second member of the right hand side

$$(n+2) \sum_{i=0}^{n-1} (i+1) \; k_{i+1} \, k_{n-i} = \left(\sum_{i=1}^{n} \frac{1}{i} \right)$$

$$- \sum_{i=0}^{n-1} \left(\frac{1}{n+3} \left[\frac{n+2}{n-i+1} - \frac{1}{i+2} \right] \right)$$

Noting that

$$\sum_{i=0}^{n-1} \frac{1}{n-i+1} = \sum_{i=2}^{n+1} \frac{1}{i} = \left(\sum_{i=1}^{n} \frac{1}{i} \right) - \frac{n}{n+1}$$

and furthermore that

$$\sum_{i=0}^{n-1} \frac{1}{i+2} = \sum_{i=2}^{n+1} \frac{1}{i} = \left(\sum_{i=1}^{n} \frac{1}{i} \right) - \frac{n}{n+1}$$

one obtains the result

$$(n+2) \sum_{i=0}^{n-1} (i+1) \; k_{i+1} \, k_{n-i} = \frac{n+2 \displaystyle\sum_{i=1}^{n} \frac{1}{i}}{n+3}$$

But, for any $n \geq 2$

$$\sum_{i=1}^{n} \frac{1}{i} \leq 1 + \frac{1}{2} + \frac{n-2}{3} = \frac{3}{2} + \frac{n-2}{3}$$

so that

$$n + 2 \sum_{i=1}^{n} \frac{1}{i} \leq \frac{5}{3} (n+1)$$

and therefore

$$\frac{n+2 \displaystyle\sum_{i=1}^{n} \frac{1}{i}}{n+3} \leq \frac{5}{3} \left(1 - \frac{2}{n+3}\right) \leq \frac{5}{3}$$

or

$$(n+2) \sum_{i=0}^{n-1} (i+1) \; k_{i+1} \, k_{n-i} \leq \frac{5}{3} \tag{48}$$

Substituting this result into (47)

$$\bar{r}\,(r_0 - \frac{5}{3}\,\bar{r}) \geq \bar{x}\,|x_0| + \bar{y}\,|y_0| + \bar{z}\,|z_0|$$

$$+ \frac{5}{3}\,(\bar{x}^2 + \bar{y}^2 + \bar{z}^2) \qquad (49)$$

To obtain a sufficient condition for (32b), we start by writing this equation as

$$(n+1)\; r_0\; |a_{1,n+1}| \leq 3(n+1)\; |r_{n+1}|\; a_{1,0}$$

$$+ \sum_{i=0}^{n-1} (i+1) \Big[3\; |r_{i+1}|\; |a_{1,n-i}| +$$

$$|a_{1,i+1}|\; |r_{n-i}| \Big]$$

Substituting (45) into this equation gives

$$(n+1)\; r_0\; |a_{1,n+1}| \leq 3\,(n+1)\; \bar{r}\; a_{1,0}\; \varepsilon^{n+1}\; k_{n+1}$$

$$+ 4\; \bar{a}_1\; \bar{r}\; \varepsilon^{n+1} \sum_{i=0}^{n-1} (i+1)\; k_{i+1}\; k_{n-i}$$

In the same way as before, a sufficient condition that (45) hold true is

$$r_0\ \bar{a}_1 \geq 3\ \bar{r}\ a_{1,0} + 4\ \bar{a}_1\ \bar{r}\ (n+2) \sum_{i=0}^{n-1} (i+1)\ k_{i+1}\ k_{n-i}$$

Incorporating the previous results for a limit on the sum of the last member of this equation gives

$$\bar{a}_1\ (r_0 - \frac{20}{3}\ \bar{r}) \geq 3\ \bar{r}\ a_{1,0} \tag{50}$$

For equation (32c)

$$(n+1)\ (n+2)\ |x_{n+2}| \leq \mu \sum_{i=0}^{n} |a_{1,i}|\ |x_{n-i}|$$

$$\leq \mu \left[a_{1,0}\ |x_n| + |a_{1,n}|\ |x_0| \right]$$

$$+ \sum_{i=1}^{n-1} |a_{1,i}|\ |x_{n-i}|$$

$$\leq \mu\ \varepsilon^n\ k_n \left[a_{1,0}\ \bar{x} + \bar{a}_1\ |x_0| \right]$$

$$+ \bar{a}_1\ \bar{x}\ \varepsilon^n \sum_{i=1}^{n-1} k_i\ k_{n-i}$$

and the sufficient condition is

$$\varepsilon^2 \bar{x} \geq \mu \frac{n+3}{n(n+1)^2} \left[a_{1,0} \bar{x} + \bar{a}_1 |x_0| \right]$$

$$+ \bar{a}_1 \bar{x} \frac{n+3}{n+1} \sum_{i=1}^{n-1} k_i k_{n-i} \tag{51}$$

We now obtain bounds on

$$\frac{n+3}{n(n+1)^2} \quad ; \quad \frac{(n+3)}{n+1} \sum_{i=1}^{n-1} k_i k_{n-i}$$

For the first of these

$$\frac{n+3}{n(n+1)^2} = \frac{n}{n(n+1)^2} + \frac{3}{n(n+1)^2}$$

$$= \frac{1}{(n+1)^2} + \frac{3}{n(n+1)^2} \leq \frac{5}{18} \qquad n \geq 2 \tag{52}$$

To obtain a bound on the remaining term, we write

$$(n+3) \sum_{i=1}^{n-1} k_i\, k_{n-i} = (n+1) \sum_{i=1}^{n-1} k_i\, k_{n-i}$$

$$+ \; 2 \sum_{i=1}^{n-1} k_i\, k_{n-i}$$

Consider the first member of the right hand side. Write

$$(n+1) = (i+1)\,(n-i+1) - i\,(n-i)$$

and therefore

$$(n+1)\; k_i\, k_{n-i} = \frac{1}{n} \left(\frac{1}{i} + \frac{1}{n-i}\right)$$

$$- \frac{1}{n+2} \left(\frac{1}{n-i+1} + \frac{1}{i+1}\right)$$

or

$$(n+1) \sum_{i=1}^{n-1} k_i \, k_{n-i} = \sum_{i=1}^{n-1} \left[\frac{1}{n} \left(\frac{1}{i} + \frac{1}{n-i} \right) \right.$$

$$\left. - \frac{1}{n+2} \left(\frac{1}{n-i+1} + \frac{1}{i+1} \right) \right]$$

$$= \frac{1}{n(n+2)} \left[\sum_{i=1}^{n-1} \left(\frac{n+2}{i} - \frac{n}{i+1} \right) \right.$$

$$\left. + \sum_{i=1}^{n-1} \left(\frac{n+2}{n-i} - \frac{n}{n-i+1} \right) \right] \tag{53}$$

Writing

$$\sum_{i=1}^{n-1} \left(\frac{n+2}{i} - \frac{n}{i+1} \right) = \left(n \sum_{i=1}^{n-1} \frac{1}{i} \right)$$

$$+ \left(2 \sum_{i=1}^{n-1} \frac{1}{i} \right) - \left(n \sum_{i=1}^{n-1} \frac{1}{i+1} \right)$$

$$= \left(2 \sum_{i=1}^{n-1} \frac{1}{i} \right) + \left(n \sum_{i=1}^{n-1} \frac{1}{i(i+1)} \right)$$

and recalling that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

We obtain

$$\sum_{i=1}^{n-1} \frac{1}{i(i+1)} = \frac{n-1}{n}$$

which yields .

$$\sum_{i=1}^{n-1} \left(\frac{n+2}{i} - \frac{n}{i+1}\right) = n - 1 + 2 \sum_{i=1}^{n-1} \frac{1}{i}$$

Consider now the last term of (53).

$$\sum_{i=1}^{n-1} \left(\frac{n+2}{n-i} - \frac{n}{n-i+1}\right) = n \sum_{i=1}^{n-1} \left(\frac{1}{n-i} - \frac{1}{n-i+1}\right)$$

$$+ 2 \sum_{i=1}^{n-1} \frac{1}{n-i}$$

Noting that

$$\sum_{i=1}^{n-1} \frac{1}{n-i} = \sum_{i=1}^{n-1} \frac{1}{i} \quad \text{and} \quad \sum_{i=1}^{n-1} \frac{1}{i+1} = \sum_{i=1}^{n-1} \frac{1}{n-i+1}$$

one obtains

$$\sum_{i=1}^{n-1} \left(\frac{n+2}{n-i} - \frac{n}{n-i+1} \right) = n-1 + 2 \sum_{i=1}^{n-1} \frac{1}{i}$$

and therefore

$$(n+1) \sum_{i=1}^{n-1} k_i \, k_{n-i} = \frac{2\left(n-1 + 2\sum_{i=1}^{n-1} \frac{1}{i}\right)}{n\,(n+2)}$$

However, for any $n \geq 2$

$$\sum_{i=1}^{n-1} \frac{1}{i} \leq n - 1$$

which gives

$$\frac{2\left(n-1 + 2 \sum_{i=1}^{n-1} \frac{1}{i}\right)}{n\ (n+2)} \leq \frac{2\ (n-1 + 2\ (n-1))}{n\ (n+2)}$$

$$= \frac{2\ (3n-3)}{n\ (n+2)} \leq \frac{6\ (n-1)}{n\ (n+2)} \leq \frac{6\ (n-1)}{n}$$

$$= 6\ (1 - \frac{1}{n}) < 6$$

and therefore

$$(n+1) \sum_{i=1}^{n-1} k_i\ k_{n-i} < 6$$

and also

$$\sum_{i=1}^{n-1} k_i\ k_{n-i} < \frac{6}{n+1} < 2 \text{ for } n \geq 2$$

Finally, we now have that

$$(n+3) \sum_{i=1}^{n-1} k_i \, k_{n-i} \leq 10 \tag{53a}$$

or

$$\frac{n+3}{n+1} \sum_{i=1}^{n-1} k_i \, k_{n-i} \leq \frac{10}{3} \quad \text{for } n \geq 2$$

Substituting this last result and (52) into (51) gives

$$\overline{x} \left(\varepsilon^2 - \frac{5}{18} \mu \, a_{1,0} - \frac{10}{3} \overline{a}_1\right) \geq \frac{5}{18} \mu \, \overline{a}_1 \, |x_0| \tag{54}$$

Symmetry gives, for (32d) and (32e)

$$\overline{y} \left(\varepsilon^2 - \frac{5}{18} \mu \, a_{1,0} - \frac{10}{3} \overline{a}_1\right) \geq \frac{5}{18} \mu \, \overline{a}_1 \, |y_0| \tag{55}$$

$$\overline{z} \left(\varepsilon^2 - \frac{5}{18} \mu \, a_{1,0} - \frac{10}{3} \overline{a}_1\right) \geq \frac{5}{18} \mu \, \overline{a}_1 \, |z_0| \tag{56}$$

Equations (49), (50), (54), (55) and (56) give the sufficient conditions for convergence of the assumed series solution. We must now show that these conditions are all compatible.

To begin with, (54), (55) and (56) can always be satisfied by choosing ε large enough and provided

$$\varepsilon^2 > + \frac{5}{18} \mu a_{1,0} + \frac{10}{3} \bar{a}_1 \tag{57}$$

Equation (50) can be satisfied provided

$$\bar{r} < \frac{3}{20} r_0 \tag{58}$$

and $\bar{a}_1$ is sufficiently large. This may require choosing ε somewhat larger for equation (57). Finally, (49) can be satisfied provided

$$\bar{r} < \frac{3}{5} r_0$$

or provided (58) is satisfied, and by choosing $\bar{x}$, $\bar{y}$, and $\bar{z}$ small enough. The smallness of $\bar{x}$, $\bar{y}$ and $\bar{z}$ in (54), (55) and (56) can always be offset by choosing a larger ε. Thus, the sufficient conditions are all compatible.

Knowing now the sufficient conditions such that all coefficients of the assumed series are bounded, we may simply follow the method of section 2.3 to show that the series are convergent.

The proof of the convergence for the partial derivative series will not be demonstrated. The results of numerical computations suggest convergence, since the coefficients for the partial series converge just as fast (numerically) as the integration series.

4.4 Numerical Applications

In order to test the accuracy of the integration scheme, the following initial conditions were chosen.

$$x = -3915.2321 \text{ km.}$$

$$y = +4802.5435 \text{ km.}$$

$$z = -3723.0849 \text{ km.}$$

$$\dot{x} = -240.95718 \text{ km./min.}$$

$$\dot{y} = -331.63944 \text{ km./min.}$$

$$\dot{z} = -169.31280 \text{ km./min.}$$

Using the value

$$\mu = 1434978970.0 \text{ km}^3/\text{min}^2$$

the corresponding two body period is

$$T = 100.5721745036 \text{ min.}$$

Twenty terms were arbitrarily taken for the assumed series solution, and the desired accuracy was $1x10^{-5}$ km. The integration step size was computed as given in section 2.4, that is, if we define C_{max} to be the largest in absolute value of x_{19}, y_{19}, and z_{19}, then

$$C_{max}\ \Delta t^{19} = 1x10^{-5}$$

or

$$\Delta t = \left[\frac{1x10^{-5}}{C_{max}}\right]^{\frac{1}{19}} \quad \text{minutes}$$

which gave values ranging between 23 and 26 minutes for the example considered here. The values of the coordinates after integrating up to the period T were

$$x_T = -3915.2321$$

$$y_T = 4802.5433$$

$$z_T = -3723.0849$$

$$\dot{x}_T = -240.95718$$

$$\dot{y}_T = -331.63946$$

$$\dot{z}_T = -169.31280$$

Using a fixed Δt of 20 minutes for 5 steps and then one step with $\Delta t = .5721745036$ yielded no differences to 8 significant digits in the initial and periodic coordinates.

Several tests are available to check the accuracy of the partial derivatives. For the first test, define a matrix ϕ to be

$$\phi = \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} & \frac{\partial x}{\partial x_1} & \frac{\partial x}{\partial y_1} & \frac{\partial x}{\partial z_1} \\ \frac{\partial y}{\partial x_0} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial z}{\partial x_0} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x}{\partial x_0} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y}{\partial x_0} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial z}{\partial x_0} & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Now subdivide ϕ so that

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

where each of the ϕ_{ij} is a 3x3 matrix. It can be shown that if the force field can be derived from a potential which is continuous, then

$$\phi^{-1} = \begin{pmatrix} \phi_{22}^T & -\phi_{12}^T \\ -\phi_{21}^T & \phi_{11}^T \end{pmatrix}$$

Then from

$$\phi \, \phi^{-1} = I$$

one obtains

$$\phi_{11} \, \phi_{22}^T - \phi_{12} \, \phi_{21}^T = I \qquad (59a)$$

$$\phi_{12} \, \phi_{11}^T - \phi_{11} \, \phi_{12}^T = 0 \qquad (59b)$$

$$\phi_{21}\,\phi_{22}^T - \phi_{22}\,\phi_{21}^T = 0 \tag{59c}$$

$$\phi_{22}\,\phi_{11}^T - \phi_{21}\,\phi_{12}^T = I \tag{59d}$$

Substitution of the computed values of the various partials into (59a) gives the following result.

$$\phi_{11}\,\phi_{21}^T - \phi_{12}\,\phi_{21}^T = \begin{pmatrix} +.999\ 999\ 87 & + & .000\ 000\ 02 & + & .000\ 000\ 09 \\ -.000\ 000\ 01 & + & .999\ 999\ 97 & - & .000\ 000\ 02 \\ +.000\ 000\ 01 & - & .000\ 000\ 03 & + & .999\ 999\ 91 \end{pmatrix}$$

A second test for the partials is to utilize the definition of a partial derivative. For example.

$$\frac{\partial x}{\partial y_1} = \frac{x\ (x_0, y_0, z_0, x_1, y_1 + \Delta y\ \ , z_1) - x\ (x_0, y_0, z_0, x_1, y_1, z_1)}{\Delta y}$$

The increment Δy was determined by taking various percentages of y_1. A sample result is shown in Table I for Δt = 10 minutes.

Table I

% y_0	$\frac{\partial x}{\partial y_1}$	Series value for $\partial x/\partial y_1$
9%	-.18102203	
5	-.18318041	
1	-.18517090	
.5	-.18540662	
.1	-.18559297	
.05	-.18561613	
.01	-.18563463	
.005	-.18563694	-.18563925
.001	-.18563879	
.0005	-.18563902	
.0001	-.18563920	
.00005	-.18563922	
.00001	-.18563925	
.000005	-.18563925	

5.0 APPLICATION TO THE ARTIFICAL SATELLITE PROBLEM WITH PERTURBATIONS DUE TO J_2, J_3, J_4

The equations defining the motion of a close earth satellite perturbed by the zonal harmonies J_2, J_3, and J_4 are

$$\ddot{x} = \frac{\partial U}{\partial x} \tag{60a}$$

$$\ddot{y} = \frac{\partial U}{\partial y} \tag{60b}$$

$$\ddot{z} = \frac{\partial U}{\partial z} \tag{60c}$$

where

$$U = \frac{\mu}{r} + \frac{J_2 \mu R^2}{2}\left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right) - \frac{J_3 \mu R^3}{2}\left(\frac{5z^3}{r^7} - \frac{3z}{r^5}\right) - \frac{3J_4 \mu R^4}{8}\left(\frac{1}{r^5} - \frac{10z^2}{r^7} + \frac{35}{3}\frac{z^4}{r^9}\right) \tag{61}$$

R = earth's equatorial radius

J_2 = second zonal harmonic

J_3 = third zonal harmonic

J_4 = fourth zonal harmonic

μ = GM = (mass earth) × (gravitational constant)

$$r^2 = x^2 + y^2 + z^2 \tag{62}$$

From (62),

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

and if we define

$$K_2 = \frac{3}{2} \mu J_2 R^2 \tag{63a}$$

$$K_3 = -\frac{1}{2} \mu J_3 R^3 \tag{63b}$$

$$K_4 = \frac{15}{8} \mu J_4 R^4 \tag{63c}$$

then,

$$\frac{\partial U}{\partial x} = -\frac{\mu x}{r^3} + K_2 \frac{x}{r^5} \left[-1 + 5 \left(\frac{z}{r}\right)^2 \right]$$

$$+ K_3 \frac{x}{r^7} \left[-35z \left(\frac{z}{r}\right)^2 + 15z \right]$$

$$+ K_4 \frac{x}{r^7} \left[1 - 14 \left(\frac{z}{r}\right)^2 + 21 \left(\frac{z}{r}\right)^4 \right]$$

$$\frac{\partial U}{\partial y} = -\frac{\mu y}{r^3} + K_2 \frac{y}{r^5} \left[-1 + 5 \left(\frac{z}{r}\right)^2 \right]$$

$$+ K_3 \frac{y}{r^7} \left[-35z \left(\frac{z}{r}\right)^2 + 15z \right]$$

$$+ K_4 \frac{y}{r^7} \left[1 - 14 \left(\frac{z}{r}\right)^2 + 21 \left(\frac{z}{r}\right)^4 \right]$$

$$\frac{\partial U}{\partial z} = -\frac{\mu z}{r^3} + K_2 \frac{z}{r^5}\left[-3 + 5\left(\frac{z}{r}\right)^2\right]$$

$$+ K_3 \frac{1}{r^5}\left[-3 + 30\left(\frac{z}{r}\right)^2 - 35\left(\frac{z}{r}\right)^4\right]$$

$$+ K_4 \frac{z}{r^7}\left[5 - \frac{70}{3}\left(\frac{z}{r}\right)^2 + 21\left(\frac{z}{r}\right)^4\right]$$

Introduce the new variables,

$$A_1 = \frac{1}{r^3} \tag{64a}$$

$$A_2 = \frac{1}{r^5} \tag{64b}$$

$$A_3 = \frac{1}{r^7} \tag{64c}$$

$$A_4 = \frac{z}{r} \tag{64d}$$

$$A_5 = \left(\frac{z}{r}\right)^2 = {A_4}^2 \tag{64e}$$

$$A_6 = \left(\frac{z}{r}\right)^4 = {A_5}^2 \tag{64f}$$

$$A_7 = x\,A_2 \tag{64g}$$

$$A_8 = x\,A_3 \tag{64h}$$

$$A_9 = y\,A_2 \tag{64i}$$

$$A_{10} = y\,A_3 \tag{64j}$$

$$A_{11} = z\,A_2 \tag{64k}$$

$$A_{12} = z\,A_3 \tag{64l}$$

$$A_{13} = z\,A_5 \tag{64m}$$

The equations to be solved are then,

$$r\dot{r} = x\dot{x} + y\dot{y} + z\dot{z} \tag{65a}$$

$$3\,A_1\,\dot{r} + r\,\dot{A}_1 = 0 \tag{65b}$$

$$5\,A_2\,\dot{r} + r\,\dot{A}_2 = 0 \tag{65c}$$

$$7 A_3 \dot{r} + r \dot{A}_3 = 0 \tag{65d}$$

$$A_4 \dot{r} + r \dot{A}_4 = \dot{z} \tag{65e}$$

$$\dot{A}_5 = 2 A_4 \dot{A}_4 \tag{65f}$$

$$\dot{A}_6 = 2 A_5 \dot{A}_5 \tag{65g}$$

$$\dot{A}_7 = x \dot{A}_2 + A_2 \dot{x} \tag{65h}$$

$$\dot{A}_8 = x \dot{A}_3 + A_3 \dot{x} \tag{65i}$$

$$\dot{A}_9 = y \dot{A}_2 + A_2 \dot{y} \tag{65j}$$

$$\dot{A}_{10} = y \dot{A}_3 + A_3 \dot{y} \tag{65k}$$

$$\dot{A}_{11} = z \dot{A}_2 + A_2 \dot{z} \tag{65l}$$

$$\dot{A}_{12} = z \dot{A}_3 + A_3 \dot{z} \tag{65m}$$

$$\dot{A}_{13} = z \dot{A}_5 + A_5 \dot{z} \tag{65n}$$

$$\ddot{x} = -\mu x A_1 + K_2 A_7 (-1 + 5 A_5)$$

$$+ 5 K_3 A_8 (3z - 7 A_{13}) \qquad (65o)$$

$$+ K_4 A_8 (1 - 14 A_5 + 21 A_6)$$

$$\ddot{y} = -\mu A_1 y + K_2 A_9 (-1 + 5 A_5)$$

$$+ 5 K_3 A_{10} (3z - 7 A_{13}) \qquad (65p)$$

$$+ K_4 A_{10} (1 - 14 A_5 + 21 A_6)$$

$$\ddot{z} = -\mu A_1 z + K_2 A_{11} (-3 + 5 A_5)$$

$$+ K_3 A_2 (-3 + 30 A_5 - 35 A_6) \qquad (65q)$$

$$+ K_4 A_{12} (5 - \frac{70}{3} A_5 + 21 A_6)$$

5.1 The Integration Formulae

Assume the following solutions:

$$x = \sum_{i=0}^{m} x_i \, \Delta t^i \tag{66a}$$

$$y = \sum_{i=0}^{m} y_i \, \Delta t^i \tag{66b}$$

$$z = \sum_{i=0}^{m} z_i \, \Delta t^i \tag{66c}$$

$$r = \sum_{i=0}^{m} r_i \, \Delta t^i \tag{66d}$$

$$A_j = \sum_{i=0}^{m} a_{j,i} \, \Delta t^i \tag{66e}$$

Then

$$\dot{x} = \sum_{i=0}^{m} (i+1) \, x_{i+1} \, \Delta t^i \tag{67a}$$

$$\dot{y} = \sum_{i=0}^{m} (i+1) \, y_{i+1} \, \Delta t^i \tag{67b}$$

$$\dot{z} = \sum_{i=0}^{m} (i+1)\ z_{i+1}\ \Delta t^i \tag{67c}$$

$$\dot{r} = \sum_{i=0}^{m} (i+1)\ r_{i+1}\ \Delta t^i \tag{67d}$$

$$\dot{A}_j = \sum_{i=0}^{m} (i+1)\ a_{j,i+1}\ \Delta t^i \tag{67e}$$

$$\ddot{x} = \sum_{i=0}^{m} (i+1)\ (i+2)\ x_{i+2}\ \Delta t^i \tag{67f}$$

$$\ddot{y} = \sum_{i=0}^{m} (i+1)\ (i+2)\ y_{i+2}\ \Delta t^i \tag{67g}$$

$$\ddot{z} = \sum_{i=0}^{m} (i+1)\ (i+2)\ z_{i+2}\ \Delta t^i \tag{67h}$$

Substituting equations (31), (66), and (67) into equations (65), we obtain the following recurrence formulae:

$$\begin{aligned}(n+1)\ r_0\ r_{n+1} = {} & (n+1)\ (x_{n+1}\ x_0 + y_{n+1}\ y_0 + z_{n+1}\ z_0) \\ & + \sum_{i=0}^{n-1} (i+1)\ (x_{i+1}\ x_{n-i} + y_{i+1}\ y_{n-i} \\ & + z_{i+1}\ z_{n-i} - r_{i+1}\ r_{n-i})\end{aligned} \tag{68a}$$

$$(n+1)\, r_0\, a_{1,n+1} = -3\,(n+1)\, a_{1,0}\, r_{n+1} - \sum_{i=0}^{n-1} (i+1)\,(a_{1,i+1}\, r_{n-i} + 3\, r_{i+1}\, a_{1,n-i}) \tag{68b}$$

$$(n+1)\, r_0\, a_{2,n+1} = -5\,(n+1)\, a_{2,0}\, r_{n+1} - \sum_{i=0}^{n-1} (i+1)\,(a_{2,i+1}\, r_{n-i} + 5\, r_{i+1}\, a_{2,n-i}) \tag{68c}$$

$$(n+1)\, r_0\, a_{3,n+1} = -7\,(n+1)\, a_{3,0}\, r_{n+1} - \sum_{i=0}^{n-1} (i+1)\,(a_{3,i+1}\, r_{n-i} + 7\, r_{i+1}\, a_{3,n-i}) \tag{68d}$$

$$(n+1)\, r_0\, a_{4,n+1} = -(n+1)\,(a_{4,0}\, r_{n+1} - z_{n+1}) - \sum_{i=0}^{n-1} (i+1)\,(a_{4,i+1}\, r_{n-i} + r_{i+1}\, a_{4,n-i}) \tag{68e}$$

$$(n+1)\ a_{5,n+1} = 2 \sum_{i=0}^{n} (i+1)\ a_{4,i+1}\ a_{4,n-i} \tag{68f}$$

$$(n+1)\ a_{6,n+1} = 2 \sum_{i=0}^{n} (i+1)\ a_{5,i+1}\ a_{5,n-i} \tag{68g}$$

$$(n+1)\ a_{7,n+1} = \sum_{i=0}^{n} (i+1)\ (a_{2,i+1}\ x_{n-i} + x_{i+1}\ a_{2,n-i}) \tag{68h}$$

$$(n+1)\ a_{8,n+1} = \sum_{i=0}^{n} (i+1)\ (a_{3,i+1}\ x_{n-i} + x_{i+1}\ a_{3,n-i}) \tag{68i}$$

$$(n+1)\ a_{9,n+1} = \sum_{i=0}^{n} (i+1)\ (a_{2,i+1}\ y_{n-i} + y_{i+1}\ a_{2,n-i}) \tag{68j}$$

$$(n+1)\ a_{10,n+1} = \sum_{i=0}^{n} (i+1)\ (a_{3,i+1}\ y_{n-i} + y_{i+1}\ a_{3,n-i}) \tag{68k}$$

$$(n+1)\ a_{11,n+1} = \sum_{i=0}^{n} (i+1)\ (a_{2,i+1}\ z_{n-i} + z_{i+1}\ a_{2,n-i}) \tag{68l}$$

$$(n+1)\ a_{12,n+1} = \sum_{i=0}^{n} (i+1)\ (a_{3,i+1}\ z_{n-i} + z_{i+1}\ a_{3,n-i}) \tag{68m}$$

$$(n+1)\ a_{13,n+1} = \sum_{i=0}^{n} (i+1)\ (a_{5,n-i}\ z_{i+1} + z_{n-i}\ a_{5,i+1}) \tag{68n}$$

$$\begin{aligned}(n+1)\ (n+2)\ x_{n+2} &= -\ K_2\ a_{7,n} + K_4\ a_{8,n} \\ &+ \sum_{i=0}^{n} \Big[-\ \mu\ a_{1,i}\ x_{n-i} + 5\ K_2\ a_{7,i}\ a_{5,n-i} \\ &+ a_{8,i} \Big\{ 7\ K_4\ (-\ 2\ a_{5,n-i} + 3\ a_{6,n-i}) \\ &+ 5\ K_3\ (3\ z_{n-i} - 7\ a_{13,n-i}) \Big\} \Big]\end{aligned} \tag{68o}$$

$$
\begin{aligned}
(n+1)\,(n+2)\, y_{n+2} &= -K_2\, a_{9,n} + K_4\, a_{10,n} \\
&+ \sum_{i=0}^{n} \Big[-\mu\, a_{1,i}\, y_{n-i} + 5 K_2\, a_{9,i}\, a_{5,n-i} \\
&+ a_{10,i} \Big\{ 7 K_4 (-2\, a_{5,n-i} + 3\, a_{6,n-i}) \\
&+ 5 K_3 (3\, z_{n-i} - 7\, a_{13,n-i}) \Big\} \Big]
\end{aligned}
\tag{68p}
$$

$$
\begin{aligned}
(n+1)\,(n+2)\, z_{n+2} &= -3\,(K_2\, a_{11,n} + K_3\, a_{2,n}) \\
&+ 5 K_4\, a_{12,n} + \sum_{i=0}^{n} \Big[-\mu\, a_{1,i}\, z_{n-i} \\
&+ 5\, a_{5,i} \Big\{ K_2\, a_{11,n-i} - \frac{14}{3} K_4\, a_{12,n-i} \\
&+ 6 K_3\, a_{2,n-i} \Big\} + 7\, a_{6,i} \Big\{ 3 K_4\, a_{12,n-i} \\
&- 5 K_3\, a_{2,n-i} \Big\} \Big]
\end{aligned}
\tag{68q}
$$

5.2 The Partial Derivative Formulae

Following the notation of section 4.2, one obtains,

0th ORDER PARTIALS

$$\frac{\partial x^*}{\partial x^*} = 1 \qquad x^* = x_0, y_0, z_0 \qquad (69a)$$

$$\frac{\partial x^*}{\partial x^{**}} = 0 \qquad \begin{matrix} x^* = x_0, y_0, z_0 \\ x^{**} = x_1, y_1, z_1 \end{matrix} \qquad (69b)$$

$$\frac{\partial r_0}{\partial x^*} = \begin{cases} \dfrac{x^*}{r_0} & x^* = x_0, y_0, z_0 \\ 0 & x^* = x_1, y_1, z_1 \end{cases} \qquad (69c)$$

$$\frac{\partial a_{1,0}}{\partial x^*} = \begin{cases} -\dfrac{3a_{1,0}}{r_0} \dfrac{\partial r_0}{\partial x^*} & x^* = x_0, y_0, z_0 \\ 0 & x^* = x_1, y_1, z_1 \end{cases} \qquad (69d)$$

In the following, x^* takes on the values x_0, y_0, z_0, x_1, y_1, and z_1.

$$\frac{\partial a_{2,0}}{\partial x^*} = -\frac{5a_{2,0}}{r_0} \frac{\partial r_0}{\partial x^*} \tag{69e}$$

$$\frac{\partial a_{3,0}}{\partial x^*} = -\frac{7a_{3,0}}{r_0} \frac{\partial r_0}{\partial x^*} \tag{69f}$$

$$\frac{\partial a_{4,0}}{\partial x^*} = \frac{1}{r_0} \frac{\partial z_0}{\partial x^*} - \frac{z_0}{r_0^2} \frac{\partial r_0}{\partial x^*} \tag{69g}$$

$$\frac{\partial a_{5,0}}{\partial x^*} = 2\, a_{4,0} \frac{\partial a_{4,0}}{\partial x^*} \tag{69h}$$

$$\frac{\partial a_{6,0}}{\partial x^*} = 2\, a_{5,0} \frac{\partial a_{5,0}}{\partial x^*} \tag{69i}$$

$$\frac{\partial a_{7,0}}{\partial x^*} = x_0 \frac{\partial a_{2,0}}{\partial x^*} + a_{2,0} \frac{\partial x_0}{\partial x^*} \tag{69j}$$

$$\frac{\partial a_{8,0}}{\partial x^*} = x_0 \frac{\partial a_{3,0}}{\partial x^*} + a_{3,0} \frac{\partial x_0}{\partial x^*} \qquad (69k)$$

$$\frac{\partial a_{9,0}}{\partial x^*} = y_0 \frac{\partial a_{2,0}}{\partial x^*} + a_{2,0} \frac{\partial y_0}{\partial x^*} \qquad (69l)$$

$$\frac{\partial a_{10,0}}{\partial x^*} = y_0 \frac{\partial a_{3,0}}{\partial x^*} + a_{3,0} \frac{\partial y_0}{\partial x^*} \qquad (69m)$$

$$\frac{\partial a_{11,0}}{\partial x^*} = z_0 \frac{\partial a_{2,0}}{\partial x^*} + a_{2,0} \frac{\partial z_0}{\partial x^*} \qquad (69n)$$

$$\frac{\partial a_{12,0}}{\partial x^*} = z_0 \frac{\partial a_{3,0}}{\partial x^*} + a_{3,0} \frac{\partial z_0}{\partial x^*} \qquad (69o)$$

$$\frac{\partial a_{13,0}}{\partial x^*} = z_0 \frac{\partial a_{5,0}}{\partial x^*} + a_{5,0} \frac{\partial z_0}{\partial x^*} \qquad (69p)$$

1st ORDER PARTIALS

$$\frac{\partial x^*}{\partial x^*} = 1 \qquad x^* = x_1,\ y_1,\ z_1 \qquad (70a)$$

$$\frac{\partial x^*}{\partial x^{**}} = 0 \qquad \begin{array}{l} x^* = x_1,\ y_1,\ z_1 \\ x^{**} = x_0,\ y_0,\ z_0 \end{array} \tag{70b}$$

$$\frac{\partial r_1}{\partial x^*} = \frac{1}{r_0}\left[x_1 \frac{\partial x_0}{\partial x^*} + x_0 \frac{\partial x_1}{\partial x^*} + y_1 \frac{\partial y_0}{\partial x^*} + y_0 \frac{\partial y_1}{\partial x^*} + z_1 \frac{\partial z_0}{\partial x^*} + z_0 \frac{\partial z_1}{\partial x^*} - r_1 \frac{\partial r_0}{\partial x^*} \right] \tag{70c}$$

$$\frac{\partial a_{1,1}}{\partial x^*} = -\frac{1}{r_0}\left[3\left(r_1 \frac{\partial a_{1,0}}{\partial x^*} + a_{1,0} \frac{\partial r_1}{\partial x^*} \right) + a_{1,1} \frac{\partial r_0}{\partial x^*} \right] \tag{70d}$$

$$\frac{\partial a_{2,1}}{\partial x^*} = - \frac{1}{r_0} \left[5 \left(r_1 \frac{\partial a_{2,0}}{\partial x^*} + a_{2,0} \frac{\partial r_1}{\partial x^*} \right) \right.$$

$$\left. + \; a_{2,1} \frac{\partial r_0}{\partial x^*} \right] \tag{70e}$$

$$\frac{\partial a_{3,1}}{\partial x^*} = - \frac{1}{r_0} \left[7 \left(r_1 \frac{\partial a_{3,0}}{\partial x^*} + a_{3,0} \frac{\partial r_1}{\partial x^*} \right) \right.$$

$$\left. + \; a_{3,1} \frac{\partial r_0}{\partial x^*} \right] \tag{70f}$$

$$\frac{\partial a_{4,1}}{\partial x^*} = - \frac{1}{r_0} \left[r_1 \frac{\partial a_{4,0}}{\partial x^*} + a_{4,0} \frac{\partial r_1}{\partial x^*} \right.$$

$$\left. - \frac{\partial z_1}{\partial x^*} + a_{4,1} \frac{\partial r_0}{\partial x^*} \right] \tag{70g}$$

$$\frac{\partial a_{5,1}}{\partial x^*} = 2 \left(a_{4,1} \frac{\partial a_{4,0}}{\partial x^*} + a_{4,0} \frac{\partial a_{4,1}}{\partial x^*} \right) \tag{70h}$$

$$\frac{\partial a_{6,1}}{\partial x^*} = 2 \left(a_{5,1} \frac{\partial a_{5,0}}{\partial x^*} + a_{5,0} \frac{\partial a_{5,1}}{\partial x^*} \right) \tag{70i}$$

$$\begin{aligned} \frac{\partial a_{7,1}}{\partial x^*} = {} & a_{2,1} \frac{\partial x_0}{\partial x^*} + x_0 \frac{\partial a_{2,1}}{\partial x^*} \\ & + a_{2,0} \frac{\partial x_1}{\partial x^*} + x_1 \frac{\partial a_{2,0}}{\partial x^*} \end{aligned} \tag{70j}$$

$$\begin{aligned} \frac{\partial a_{8,1}}{\partial x^*} = {} & a_{3,1} \frac{\partial x_0}{\partial x^*} + x_0 \frac{\partial a_{3,1}}{\partial x^*} \\ & + a_{3,0} \frac{\partial x_1}{\partial x^*} + x_1 \frac{\partial a_{3,0}}{\partial x^*} \end{aligned} \tag{70k}$$

$$\frac{\partial a_{9,1}}{\partial x^*} = a_{2,1} \frac{\partial y_0}{\partial x^*} + y_0 \frac{\partial a_{2,1}}{\partial x^*}$$

$$+ a_{2,0} \frac{\partial y_1}{\partial x^*} + y_1 \frac{\partial a_{2,0}}{\partial x^*} \tag{70l}$$

$$\frac{\partial a_{10,1}}{\partial x^*} = a_{3,1} \frac{\partial y_0}{\partial x^*} + y_0 \frac{\partial a_{3,1}}{\partial x^*}$$

$$+ a_{3,0} \frac{\partial y_1}{\partial x^*} + y_1 \frac{\partial a_{3,0}}{\partial x^*} \tag{70m}$$

$$\frac{\partial a_{11,1}}{\partial x^*} = a_{2,1} \frac{\partial z_0}{\partial x^*} + z_0 \frac{\partial a_{2,1}}{\partial x^*}$$

$$+ a_{2,0} \frac{\partial z_1}{\partial x^*} + z_1 \frac{\partial a_{2,0}}{\partial x^*} \tag{70n}$$

$$\frac{\partial a_{12,1}}{\partial x^*} = a_{3,1} \frac{\partial z_0}{\partial x^*} + z_0 \frac{\partial a_{3,1}}{\partial x^*} + a_{3,0} \frac{\partial z_1}{\partial x^*} + z_1 \frac{\partial a_{3,0}}{\partial x^*} \tag{70o}$$

$$\frac{\partial a_{13,1}}{\partial x^*} = a_{5,0} \frac{\partial z_1}{\partial x^*} + z_1 \frac{\partial a_{5,0}}{\partial x^*} + a_{5,1} \frac{\partial z_0}{\partial x^*} + z_0 \frac{\partial a_{5,1}}{\partial x^*} \tag{70p}$$

nth ORDER PARTIALS

$$(n+1)\ r_0\ \frac{\partial r_{n+1}}{\partial x^*} = (n+1)\left[x_{n+1}\ \frac{\partial x_0}{\partial x^*} + x_0\ \frac{\partial x_{n+1}}{\partial x^*} \right.$$

$$+ y_{n+1}\ \frac{\partial y_0}{\partial x^*} + y_0\ \frac{\partial y_{n+1}}{\partial x^*} + z_{n+1}\ \frac{\partial z_0}{\partial x^*}$$

$$\left. + z_0\ \frac{\partial z_{n+1}}{\partial x^*} - r_{n+1}\ \frac{\partial r_0}{\partial x^*} \right] \tag{71a}$$

$$+ \sum_{i=0}^{n-1} (i+1) \left\{ x_{i+1}\ \frac{\partial x_{n-i}}{\partial x^*} + x_{n-i}\ \frac{\partial x_{i+1}}{\partial x^*} \right.$$

$$+ y_{i+1}\ \frac{\partial y_{n-i}}{\partial x^*} + y_{n-i}\ \frac{\partial y_{i+1}}{\partial x^*} + z_{i+1}\ \frac{\partial z_{n-i}}{\partial x^*}$$

$$\left. + z_{n-i}\ \frac{\partial z_{i+1}}{\partial x^*} - r_{i+1}\ \frac{\partial r_{n-i}}{\partial x^*} - r_{n-i}\ \frac{\partial r_{i+1}}{\partial x^*} \right\}$$

$$
\begin{aligned}
(n+1)\ r_0\ \frac{\partial a_{1,n+1}}{\partial x^*} = -\ (n+1)\Bigg[\ & a_{1,n+1}\ \frac{\partial r_0}{\partial x^*} \\
& + 3\left(r_{n+1}\ \frac{\partial a_{1,0}}{\partial x^*} + a_{1,0}\ \frac{\partial r_{n+1}}{\partial x^*}\right)\Bigg] \\
& - \sum_{i=0}^{n-1} (i+1) \Bigg\{ 3\left(r_{i+1}\ \frac{\partial a_{1,n-i}}{\partial x^*}\right. \\
& \left. + a_{1,n-i}\ \frac{\partial r_{i+1}}{\partial x^*}\right) \\
& + a_{1,i+1}\ \frac{\partial r_{n-i}}{\partial x^*} + r_{n-i}\ \frac{\partial a_{1,i+1}}{\partial x^*} \Bigg\}
\end{aligned}
\tag{71b}
$$

$$
\begin{aligned}
(n+1) \quad r_0 \frac{\partial a_{2,n+1}}{\partial x^*} = - (n+1) \Bigg[a_{2,n+1} \frac{\partial r_0}{\partial x^*} \\
+ 5 \left(a_{2,0} \frac{\partial r_{n+1}}{\partial x^*} + r_{n+1} \frac{\partial a_{2,0}}{\partial x^*} \right) \Bigg] \\
- \sum_{i=0}^{n-1} (i+1) \Bigg\{ a_{2,i+1} \frac{\partial r_{n-i}}{\partial x^*} \\
+ r_{n-i} \frac{\partial a_{2,i+1}}{\partial x^*} + 5 \left(r_{i+1} \frac{\partial a_{2,n-i}}{\partial x^*} \right. \\
\left. + a_{2,n-i} \frac{\partial r_{i+1}}{\partial x^*} \right) \Bigg\}
\end{aligned}
\tag{71c}
$$

$$
\begin{aligned}
(n+1)\ r_0 \frac{\partial a_{3,n+1}}{\partial x^*} = -(n+1) \Bigg[& a_{3,n+1} \frac{\partial r_0}{\partial x^*} \\
& + 7 \left(a_{3,0} \frac{\partial r_{n+1}}{\partial x^*} + r_{n+1} \frac{\partial a_{3,0}}{\partial x^*} \right) \Bigg] \\
& - \sum_{i=0}^{n-1} (i+1) \Bigg\{ a_{3,i+1} \frac{\partial r_{n-i}}{\partial x^*} \\
& + r_{n-i} \frac{\partial a_{3,i+1}}{\partial x^*} + 7 \left(r_{i+1} \frac{\partial a_{3,n-i}}{\partial x^*} \right. \\
& \left. + a_{3,n-i} \frac{\partial r_{i+1}}{\partial x^*} \right) \Bigg\}
\end{aligned}
\qquad (71d)
$$

$$(n+1) \quad r_0 \frac{\partial a_{4,n+1}}{\partial x^*} = -(n+1)\left[a_{4,n+1} \frac{\partial r_0}{\partial x^*} - \frac{\partial z_{n+1}}{\partial x^*}\right.$$

$$\left. a_{4,0} \frac{\partial r_{n+1}}{\partial x^*} + r_{n+1} \frac{\partial a_{4,0}}{\partial x^*} \right] \qquad (71e)$$

$$- \sum_{i=0}^{n-1} (i+1) \left\{ a_{4,i+1} \frac{\partial r_{n-i}}{\partial x^*} + r_{n-i} \frac{\partial a_{4,i+1}}{\partial x^*} \right.$$

$$\left. + r_{i+1} \frac{\partial a_{4,n-i}}{\partial x^*} + a_{4,n-i} \frac{\partial r_{i+1}}{\partial x^*} \right\}$$

$$(n+1) \frac{\partial a_{5,n+1}}{\partial x^*} = 2 \sum_{i=0}^{n} (i+1) \left\{ a_{4,i+1} \frac{\partial a_{4,n-i}}{\partial x^*} \right.$$

$$\left. + a_{4,n-i} \frac{\partial a_{4,i+1}}{\partial x^*} \right\} \qquad (71f)$$

$$(n+1) \frac{\partial a_{6,n+1}}{\partial x^*} = 2 \sum_{i=0}^{n} (i+1) \left\{ a_{5,i+1} \frac{\partial a_{5,n-i}}{\partial x^*} \right.$$

$$\left. + \; a_{5,n-i} \frac{\partial a_{5,i+1}}{\partial x^*} \right\} \tag{71g}$$

$$(n+1) \frac{\partial a_{7,n+1}}{\partial x^*} = \sum_{i=0}^{n} (i+1) \left\{ a_{2,i+1} \frac{\partial x_{n-i}}{\partial x^*} \right.$$

$$+ \; x_{n-i} \frac{\partial a_{2,i+1}}{\partial x^*} + x_{i+1} \frac{\partial a_{2,n-i}}{\partial x^*} \tag{71h}$$

$$\left. + \; a_{2,n-i} \frac{\partial x_{i+1}}{\partial x^*} \right\}$$

$$(n+1)\frac{\partial a_{8,n+1}}{\partial x^*} = \sum_{i=0}^{n}(i+1)\left\{ a_{3,i+1}\frac{\partial x_{n-i}}{\partial x^*} + x_{n-i}\frac{\partial a_{3,i+1}}{\partial x^*} + x_{i+1}\frac{\partial a_{3,n-i}}{\partial x^*} + a_{3,n-i}\frac{\partial x_{i+1}}{\partial x^*}\right\} \tag{71i}$$

$$(n+1)\frac{\partial a_{9,n+1}}{\partial x^*} = \sum_{i=0}^{n}(i+1)\left\{ a_{2,i+1}\frac{\partial y_{n-i}}{\partial x^*} + y_{n-i}\frac{\partial a_{2,i+1}}{\partial x^*} + y_{i+1}\frac{\partial a_{2,n-i}}{\partial x^*} + a_{2,n-i}\frac{\partial y_{i+1}}{\partial x^*}\right\} \tag{71j}$$

$$(n+1)\frac{\partial a_{10,n+1}}{\partial x^*} = \sum_{i=0}^{n}(i+1)\left\{a_{3,i+1}\frac{\partial y_{n-i}}{\partial x^*}\right.$$

$$+ y_{n-i}\frac{\partial a_{3,i+1}}{\partial x^*} + y_{i+1}\frac{\partial a_{3,n-i}}{\partial x^*} \tag{71k}$$

$$\left. + a_{3,n-i}\frac{\partial y_{i+1}}{\partial x^*}\right\}$$

$$(n+1)\frac{\partial a_{11,n+1}}{\partial x^*} = \sum_{i=0}^{n}(i+1)\left\{a_{2,i+1}\frac{\partial z_{n-i}}{\partial x^*}\right.$$

$$+ z_{n-i}\frac{\partial a_{2,i+1}}{\partial x^*} + z_{i+1}\frac{\partial a_{2,n-i}}{\partial x^*} \tag{71l}$$

$$\left. + a_{2,n-i}\frac{\partial z_{i+1}}{\partial x^*}\right\}$$

$$(n+1)\,\frac{\partial a_{12,n+1}}{\partial x^*} = \sum_{i=0}^{n} (i+1) \left\{ a_{3,i+1}\,\frac{\partial z_{n-i}}{\partial x^*} \right.$$

$$+\; z_{n-i}\,\frac{\partial a_{3,i+1}}{\partial x^*} \;+\; z_{i+1}\,\frac{\partial a_{3,n-i}}{\partial x^*} \qquad (71m)$$

$$\left. +\; a_{3,n-i}\,\frac{\partial z_{i+1}}{\partial x^*} \right\}$$

$$(n+1)\,\frac{\partial a_{13,n+1}}{\partial x^*} = \sum_{i=0}^{n} (i+1) \left\{ a_{5,n-i}\,\frac{\partial z_{i+1}}{\partial x^*} \right.$$

$$+\; z_{i+1}\,\frac{\partial a_{5,n-i}}{\partial x^*} \;+\; z_{n-i}\,\frac{\partial a_{5,i+1}}{\partial x^*} \qquad (71n)$$

$$\left. +\; a_{5,i+1}\,\frac{\partial z_{n-i}}{\partial x^*} \right\}$$

$$
\begin{aligned}
(n+1)\,(n+2)\,\frac{\partial x_{n+2}}{\partial x^*} = & - K_2 \frac{\partial a_{7,n}}{\partial x^*} + K_4 \frac{\partial a_{8,n}}{\partial x^*} \\
& + \sum_{i=0}^{n} \Bigg[- \mu \left(a_{1,i} \frac{\partial x_{n-i}}{\partial x^*} + x_{n-i} \frac{\partial a_{1,i}}{\partial x^*} \right) \\
& + 5 K_2 \left(a_{7,i} \frac{\partial a_{5,n-i}}{\partial x^*} + a_{5,n-i} \frac{\partial a_{7,i}}{\partial x^*} \right) \\
& + a_{8,i} \Bigg\{ 7 K_4 \left(-2 \frac{\partial a_{5,n-i}}{\partial x^*} + 3 \frac{\partial a_{6,n-i}}{\partial x^*} \right) \\
& + 5 K_3 \left(3 \frac{\partial z_{n-i}}{\partial x^*} - 7 \frac{\partial a_{13,n-i}}{\partial x^*} \right) \Bigg\} \\
& + \frac{\partial a_{8,i}}{\partial x^*} \Bigg\{ 7 K_4 \left(-2\, a_{5,n-i} + 3\, a_{6,n-i} \right) \\
& + 5 K_3 \left(3\, z_{n-i} - 7\, a_{13,n-i} \right) \Bigg\} \Bigg]
\end{aligned}
\tag{71o}
$$

$$
\begin{aligned}
(n+1)(n+2)\frac{\partial y_{n+2}}{\partial x^*} &= -K_2\frac{\partial a_{9,n}}{\partial x^*} + K_4\frac{\partial a_{10,n}}{\partial x^*} \\
&+ \sum_{i=0}^{n}\Bigg[-\mu\left(a_{1,i}\frac{\partial y_{n-i}}{\partial x^*} + y_{n-i}\frac{\partial a_{1,i}}{\partial x^*}\right) \\
&+ 5K_2\left(a_{9,i}\frac{\partial a_{5,n-i}}{\partial x^*} + a_{5,n-i}\frac{\partial a_{9,i}}{\partial x^*}\right) \\
&+ a_{10,i}\Bigg\{7K_4\left(-2\frac{\partial a_{5,n-i}}{\partial x^*} + 3\frac{\partial a_{6,n-i}}{\partial x^*}\right) \\
&+ 5K_3\left(3\frac{\partial z_{n-i}}{\partial x^*} - 7\frac{\partial a_{13,n-i}}{\partial x^*}\right)\Bigg\} \qquad (71p) \\
&+ \frac{\partial a_{10,i}}{\partial x^*}\Bigg\{7K_4\left(-2a_{5,n-i} + 3a_{6,n-i}\right) \\
&+ 5K_3\left(3z_{n-i} - 7a_{13,n-i}\right)\Bigg\}\Bigg]
\end{aligned}
$$

$$(n+1)\,(n+2)\,\frac{\partial z_{n+2}}{\partial x^*} = -3\left(K_2\,\frac{\partial a_{11,n}}{\partial x^*} + K_3\,\frac{\partial a_{2,n}}{\partial x^*}\right)$$

$$+ 5\,K_4\,\frac{\partial a_{12,n}}{\partial x^*} + \sum_{i=0}^{n}\left[-\mu\left(a_{1,i}\,\frac{\partial z_{n-i}}{\partial x^*}\right.\right.$$

$$\left. + z_{n-i}\,\frac{\partial a_{1,i}}{\partial x^*}\right) + 5\,a_{5,i}\left\{K_2\,\frac{\partial a_{11,n-i}}{\partial x^*}\right.$$

$$\left. - \frac{14}{3}\,K_4\,\frac{\partial a_{12,n-i}}{\partial x^*} + 6\,K_3\,\frac{\partial a_{2,n-i}}{\partial x^*}\right\} \tag{71q}$$

$$+ 5\,\frac{\partial a_{5,i}}{\partial x^*}\left\{K_2\,a_{11,n-i} - \frac{14}{3}\,K_4\,a_{12,n-i}\right.$$

$$\left. + 6\,K_3\,a_{2,n-i}\right\} + 7\,a_{6,i}\left\{3\,K_4\,\frac{\partial a_{12,n-i}}{\partial x^*}\right.$$

$$\left. - 5\,K_3\,\frac{\partial a_{2,n-i}}{\partial x^*}\right\} + 7\,\frac{\partial a_{6,i}}{\partial x^*}\left\{3\,K_4\,a_{12,n-i}\right.$$

$$\left.\left. - 5\,K_3\,a_{2,n-i}\right\}\right]$$